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Classical Effects from a Factorized Spinor Representation of Maxwell's Equations

MENDEL SACHS

Department of Physics, State University of New York, Buffalo, New York

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Abstract

It was demonstrated in earlier work that the vector representation of electromagnetic theory can be factorized into a pair of two-component spinor field equations (Sachs & Schwebel, 1962). The latter is a generalization of the usual formalism, in the sense that in addition to predicting all of the effects that are implied by the vector theory, it predicts additional observable effects that are out of the domain of prediction of the Maxwell formalism. The latter extra predictions were derived in previous publications (Sachs & Schwebel, 1961, 1963; Sachs, 1968a, b). In this paper, the spinor formalism is applied to effects that are expected to agree with the predictions of the standard formalism—the Coulomb force between point charges and the measured speed of a charged particle which moves in an electric potential. While there are no vector or tensor variables involved in this formalism, the results are found, as expected, to be in agreement with the conventional representation of electromagnetic theory. The analysis serves the role of demonstrating that in the appropriate limiting case, the factorized spinor formulation of electromagnetism does predict the explicit classical effects that are also predicted by Maxwell's field equations. The paper also presents a derivation of the general form of the solutions of the spinor field equations.

1. *Introduction*

Electromagnetic field theory led to the first discovered set of laws that were consistent with the theory of special (and general) relativity. Indeed, the form of Maxwell's equations provided a seminal influence for Einstein's theoretical development of the relativity concept, from the point of view of its philosophical content as well as its mathematical structure.

The essential starting point for Einstein's theory is the principle of relativity—the assertion that the laws of nature that describe any sort of physical phenomena must have the same form in any frame of reference that is distinguishable from any other in terms of their relative motion. Since the description of motion entails a minimum of four parameters—which we may associate with the space-time coordinates (or sometimes with the energy-momentum coordinates, e.g. in scattering problems)—the principle of relativity is the assertion of a symmetry principle in a four-dimensional space-time. The symmetry group that underlies this theory is then represented by that set of transformations between the space-time coordinates of one observer and any other observer who is moving relative to the first, such that the forms of the laws of nature that they deduce in their respective frames of reference are the same.

When the relative motion is of an arbitrary type (i.e. general relativity) the irreducible representations of this group are in terms of the sixteen parameters $\{\partial x^{\mu'}/\partial x^{\nu}\}$ which are, in turn, a function of the space-time points where they are evaluated. In this general case, then, the space-time coordinate system is non-linear. The underlying group for this symmetry principle (which will be referred to as the 'Einstein group') is then a sixteen-parameter Lie group. It is a 'Lie group' because of further physical assumptions that require that the transformations be analytic as well as continuous. In the special case where the relative motion corresponds to constant rectilinear speed (i.e., special relativity) the Einstein group reduces to the ten-parameter group called the Poincaré group. In the latter limit the space-time is linear and the constant parameters $\{\partial x^{\mu'}/\partial x^{\nu}\}$ are the three Eulerian angles—describing the space $\leftrightarrow$ space transformations, the three components of constant speed, v_i/c, describing the space $\leftrightarrow$ time transformations and the four translations in space and time. In this paper, reference will only be made to the case of special relativity, although the results do carry over to the general case.

When one considers that the Maxwell field equations are a special representation of the Poincaré group, then the following question naturally arises: Is this the most primitive representation of the group? In answer to this question, it has been shown in an earlier publication (Sachs & Schwebel, 1962) that indeed, when one takes into account the fact that the principle of relativity entails only relative motion—implying that the underlying group is a continuous parameter group—the usual vector representation of the Maxwell theory factorizes into a pair of uncoupled first-rank spinor equations. The extension of this result to the case of general relativity was then carried out (Sachs, 1964).

The description of electromagnetic phenomena in terms of a spinor language has a substantial literature.† However, these earlier works did not generalize the Maxwell formalism. They rather re-expressed one four-dimensional representation (the usual vector-tensor form) in terms of another four-dimensional representation—one that entails a second-rank spinor field. Of course, this is always possible since the second-rank spinor is an entity that is in one-to-one correspondence with the four-vector. The latter was a reformulation that can serve quite usefully in solving problems in electromagnetic theory.

On the other hand, the factorization of the Maxwell formalism into a pair of first-rank spinor field equations leads to a generalization in the sense that the new formalism makes more predictions of physical observables

† See, for example, Laporte, O. and Uhlenbeck, G. E. (1931). *Physical Review*, **37**, 1380; Oppenheimer, J. R. (1931). *Physical Review*, **38**, 725; Moliere, G. (1949). *Annalen der Physik*, **6**, 146; Ohmura, T. (1956). *Progress of Theoretical Physics* (*Kyoto*), **16**, 684; and Moses, H. E. (1959). *Physical Review*, **113**, 1670.

than does the vector formalism. *Some* of these predictions are in one-to-one correspondence with *all* of the physical predictions of the vector representation of the theory. But the remaining predictions of the spinor theory have no counterpart in the vector theory. This is analogous to a consequence of Dirac's factorization of the Klein–Gordon equation, leading, for example, to the energy coupling term $\boldsymbol{\sigma}.\mathbf{H}$, which has no counterpart in the scalar formalism.

Most of this author's applications of the factorized spinor form of the electromagnetic equations have been made in the description of microscopic physics. This is because it is only in this domain where the additional predictions that have no counterpart in the Maxwell theory show up. These applications have had to do with the fine structure of hydrogen (Sachs & Schwebel, 1961), pair annihilation and creation (Sachs, 1968a), electron-proton scattering (Sachs & Schwebel, 1963) and electron-alpha scattering (Sachs, 1968b). The reason that these additional predictions do not play any role in problems concerned with macroscopic physics, or in the energy region where the Schrödinger wave equation is a valid representation of quantum mechanics, is that they have to do with interaction terms that mix the spinor components of the Dirac field. In the low-energy limit, the multi-component Dirac field reduces to the single-component Schrödinger field so that the effect of mixing spinor components automatically vanishes.

Even though the factorized spinor representation of electromagnetic theory has been shown, generally, to make the same predictions as the usual representation of Maxwell theory, because of the one-to-one correspondence between all of the conservation laws of the vector theory and some of those of the spinor theory, it still may seem peculiar and hard to understand this generalized representation of electromagnetic theory to one who is used to thinking about electromagnetic phenomena in terms of matching the vector and tensor field variables with *forces*, since there are no vector or tensor field variables here at all. Thus, to facilitate further an understanding of the physical meaning of the spinor field variables in electromagnetic theory, as well as demonstrating the mathematical solutions of the spinor field equations, the theory will be applied in this paper to explicit problems in the macroscopic domain—where one should expect identical predictions with those of the conventional vector representation of the theory. The primary purpose in this paper is then to determine the general form of the spinor field solutions and then to apply these to the derivation of two classical electromagnetic effects: (1) the Coulomb force that is exerted by one point charge on another, and (2) the measured velocity of a charged particle.

2. *The Factorized Spinor Equations and their Solutions*

To demonstrate the factorization of the Maxwell field equations into a pair of first-rank spinor equations, an initial identification can be made between the real variables **E**, **H** of the vector representation, and the complex components of the variables of the spinor representation. Thus, consider the complex vector G_μ, whose space and time components are

$$G_k = (\mathbf{H} + i\mathbf{E})_k, \qquad G_0 = 0 \qquad (k = 1, 2, 3) \tag{2.1}$$

and let the structuring of the two-component spinor variables be guided by the following correspondence between the components of a quaternion and those of a four-vector

$$\begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{2.2}$$

With this correspondence, an initial guess at the identification between the components of the electromagnetic spinor field solutions and sources and those of the Maxwell formalism is the following

$$\begin{aligned} \varphi_1 &= \begin{pmatrix} G_3 \\ G_1 + iG_2 \end{pmatrix}, \qquad & Y_1 &= -4\pi i \begin{pmatrix} \rho + j_3 \\ j_1 + ij_2 \end{pmatrix} \\ \varphi_2 &= \begin{pmatrix} G_1 - iG_2 \\ -G_3 \end{pmatrix}, \qquad & Y_2 &= -4\pi i \begin{pmatrix} j_1 - ij_2 \\ \rho - j_3 \end{pmatrix} \end{aligned} \tag{2.3}$$

It is readily verified with the quaternion differential operator defined as follows

$$\sigma_\mu \partial^\mu = \sigma_0 \partial^0 - \boldsymbol{\sigma} . \boldsymbol{\nabla} = \begin{pmatrix} \partial^0 - \partial^3 & -(\partial^1 - i\partial^2) \\ -(\partial^1 + i\partial^2) & \partial^0 + \partial^3 \end{pmatrix} \tag{2.4a}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.4b}$$

are the Pauli matrices, that the two, two-component spinor equations

$$\sigma_\mu \partial^\mu \varphi_\alpha = Y_\alpha \qquad (\alpha = 1, 2) \tag{2.5}$$

(corresponding to four complex or eight real equations) are in one-to-one correspondence with the eight real equations of the Maxwell theory

$$\begin{aligned} \partial^0 \mathbf{H} + \boldsymbol{\nabla} \times \mathbf{E} = 0 \qquad & \boldsymbol{\nabla} \times \mathbf{H} - \partial^0 \mathbf{E} = 4\pi \mathbf{j} \\ \boldsymbol{\nabla} . \mathbf{H} = 0 \qquad & \boldsymbol{\nabla} . \mathbf{E} = 4\pi\rho \\ (c = 1) & \end{aligned} \tag{2.6}$$

It is important to note at this stage that the relation (2.3) between the solutions of the spinor equations (2.5) and the Maxwell field equations (2.6) is strictly an identification in a given Lorentz frame. Under the space-time transformations of special relativity (the Poincaré group) the transformed variables $\varphi_\alpha'(x')$, $Y_\alpha'(x')$ cannot be form-invariant with respect to the field variables **E**, **H**. This is because φ_α, Υ_α are the basis functions of the two-dimensional representations of the Poincaré group, while **E**, **H** belong to the four-dimensional representations of this group. This lack of equivalence is obvious from the fact that the spinor transformations S are two-dimensional matrices that would mix the components G_k in the initial identification in equation (2.3) (Sachs & Schwebel, 1962).

Nevertheless, the *physical requirement* of the theory demands that a form-invariant correspondence must persist in the invariants and conservation equations of the respective formulations. This, of course, is because it is the conservation equations, and not the field equations themselves, that relate directly to the measured values of the properties of charged matter—and the *predictions* of the vector form of the theory certainly have empirical validity. Such a correspondence between some of the invariants and conservation equations of the factorized spinor form of electromagnetism, and all of those of Maxwell theory, was demonstrated in the earlier publications (Sachs & Schwebel, 1962; Sachs, 1964).

Once this correspondence is established in any frame of reference, the procedure is to consider the spinor field equations (2.5) to represent electromagnetism—without at all using the variables of the vector representation of the theory. That is to say, all electromagnetic phenomena should now be predictable from the solutions, $\varphi_1(x)$, $\varphi_2(x)$ of these field equations.

2.1. *Conservation Equations*

Multiplying equation (2.5) on the left with hermitian conjugate of φ_β, we have

$$\varphi_\beta^\dagger \sigma_\mu \partial^\mu \varphi_\alpha = \varphi_\beta^\dagger \Upsilon_\alpha$$

Taking the hermitian adjoint of this equation and interchanging the labels (α, β), we have

$$\partial^\mu \varphi_\beta^\dagger \sigma_\mu \varphi_\alpha = \Upsilon_\beta^\dagger \varphi_\alpha$$

Adding these two equations, we obtain the following four (complex) conservation equations

$$\partial^\mu(\varphi_\beta^\dagger \sigma_\mu \varphi_\alpha) = (\varphi_\beta^\dagger \Upsilon_\alpha + \Upsilon_\beta^\dagger \varphi_\alpha) \tag{2.1.1}$$

These correspond to *eight* real conservation equations, as contrasted with the *four* real conservation equations

$$\partial_\mu T_\nu{}^\mu = k_\nu \tag{2.1.2}$$

of the vector-tensor formalism. Here $T_\nu{}^\mu$ is the electromagnetic energy-momentum tensor and k_ν is the four-Lorentz force density

$$k_\nu = \{\mathbf{k}; k_0\} = \{\rho\mathbf{E} + \mathbf{j} \times \mathbf{H}; -\mathbf{j}\cdot\mathbf{E}\} \tag{2.1.3}$$

To exhibit the correspondence between some of the conservation equations (2.1.1) and the standard ones (2.1.2), consider the sum of equation (2.1.1) with $\alpha = \beta = 1$ and equation (2.1.1) with $\alpha = \beta = 2$. It is readily verified, by direct substitution of the identification (2.3) with the standard variables, that this sum does indeed correspond with the standard energy conservation equation that relates to Poynting's equation, i.e.,

$$-\frac{1}{16\pi}\,\partial^\mu(\varphi_1{}^\dagger\sigma_\mu\varphi_1+\varphi_2{}^\dagger\sigma_\mu\varphi_2)=-\frac{1}{16\pi}(\varphi_1{}^\dagger Y_1+\varphi_2{}^\dagger Y_2+\text{h.c.})$$

$$\Leftrightarrow\ \frac{1}{8\pi}\,\partial^0(E^2+H^2)+\frac{1}{4\pi}\,\nabla.(\mathbf{E}\times\mathbf{H})=-\mathbf{E}.\mathbf{j} \tag{2.1.4}$$

Similarly, three other real equations [equation (3.1)] correspond to the momentum conservation equations of electromagnetic theory. But there are four more conservation equations with no counterpart in the vector-tensor formalism. These will not be discussed in this paper. Some of them relate to new predictions that were derived in earlier publications (Sachs & Schwebel, 1961, 1963; Sachs, 1968a, b).

It should be emphasized at this point that a generalization which occurs here lies in the feature that the right-hand side of equation (2.1.1)—the terms that play the role in spinor theory of the Lorentz force density—are *separately* relativistically invariant. This follows from the covariance properties of equation (2.5). It was shown earlier (Sachs & Schwebel, 1962) that if the transformations of the Poincaré group induce the change in the spinor variables

$$\varphi_\alpha(x)\ \rightarrow\ \varphi_\alpha'(x')=S\varphi_\alpha(x) \tag{2.1.5a}$$

then equation (2.5) will be relativistically covariant if and only if the source fields transform as follows

$$Y_\alpha(x)\ \rightarrow\ Y_\alpha'(x')=(S^\dagger)^{-1}Y_\alpha(x) \tag{2.1.5b}$$

where S solves the equation

$$S^\dagger\sigma_\mu S=(\partial x_\mu'/\partial x_\nu)\,\sigma_\nu \tag{2.1.5c}$$

Thus, any of the combinations on the right-hand side of equation (2.1.1) transform as a scalar

$$\varphi_\alpha{}^\dagger Y_\beta\ \rightarrow\ \varphi_\alpha{}^{\dagger\prime}Y_\beta'=\varphi_\alpha{}^\dagger S^\dagger(S^\dagger)^{-1}Y_\beta=\varphi_\alpha{}^\dagger Y_\beta \tag{2.1.6}$$

This result is in contrast with the conventional formalism which yields a law of energy-momentum conservation—a single entity with four components that cannot be uncoupled when comparing the observations of electromagnetic phenomena in different inertial frames.

2.2 *Solutions of the Spinor Field Equations* (2.5)

For the problems under consideration in this article, it will be necessary to determine the particular solutions of the spinor field equations (2.5). This will be done by using the method of Fourier transforms. Substituting the Fourier integral

$$\varphi_\alpha(x)=\int\varphi_\alpha(k)\exp(ik^\mu x_\mu)\,d^4k \tag{2.2.1}$$

into the field equations (2.5), we have

$$\sigma_\mu \partial^\mu \int \varphi_\alpha(k) \exp(ik^\mu x_\mu)\, d^4 k = \int (i\sigma_\mu k^\mu)\, \varphi_\alpha(k) \exp(ik^\mu x_\mu)\, d^4 k = Y_\alpha$$

Multiplying both sides of this equation by the conjugate quaternion

$$(-ik^\rho \bar{\sigma}_\rho) \exp(-ik^{\mu'} x_\mu)$$

where $\bar{\sigma}_\rho = \{\sigma_0; -\boldsymbol{\sigma}\}$ are the conjugate quaternion basis elements, and integrating over space-time, we have

$$\int K(k', k)\, \varphi_\alpha(k) \exp[i(k^\mu - k^{\mu'})\, x_\mu]\, d^4 x\, d^4 k$$
$$= -i \int k^\rho \bar{\sigma}_\rho Y_\alpha(x) \exp(-ik^{\mu'} x_\mu)\, d^4 x \quad (2.2.2)$$

where

$$K(k, k') = (k^{\rho'} \bar{\sigma}_\rho)(k^\mu \sigma_\mu) \equiv k^{\rho'} k_\rho - \sigma_0 \sigma_j (k^{0'} k^j - k^{j'} k^0) - \\ - \sigma_j \sigma_m (k^{j'} k^m - k^{m'} k^j)$$

and

$$k^{\rho'} k_\rho = k^{0'} k^0 - \mathbf{k}' . \mathbf{k}$$

Using the integral representation of the Dirac delta function

$$\int \exp[i(k^{\mu'} - k^\mu)\, x_\mu]\, d^4 x = (2\pi)^4\, \delta(k^{\mu'} - k^\mu)$$

in the left side of equation (2.2.2) and the property that $K(k,k) = k^\nu k_\nu$, we have the Fourier coefficient

$$\varphi_\alpha(k) = \frac{-i}{(2\pi)^4} \int \frac{(k^\rho \bar{\sigma}_\rho)}{k^\nu k_\nu} Y_\alpha(x') \exp(-ik^\mu x_\mu')\, d^4 x' \quad (2.2.3)$$

Substituting equation (2.2.3) into equation (2.2.1), the particular solutions of the spinor field equations (2.5) then take the form

$$\varphi_\alpha(x) = \int \left\{ \left[\frac{(-i\bar{\sigma}_\rho k^\rho)}{(2\pi)^4 k^\nu k_\nu} \exp[ik^\mu (x_\mu - x_\mu')] \right] d^4 k \right\} Y_\alpha(x')\, d^4 x' \quad (2.2.4)$$

Note that the quaternion function in the square bracket, integrated over k-space, is the Green's function for the field equations (2.5).

2.3. *Spinor Field Solutions for a Static Point Charge*

Starting with the structure of the source field variables Y_α that correspond to a static point charge (in any frame of reference†) at the origin, we have, according to the identification in equation (2.3),

$$Y_1 = -4\pi i e \delta(\mathbf{r}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad Y_2 = -4\pi i e \delta(\mathbf{r}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.3.1)$$

† Note that the assumption of a fixed origin in this application is based on the non-relativistic approximation which assumes that the point charge at the origin absorbs a negligible amount of energy and momentum from its environment, so that it can be considered at rest for all times, if it is initially in this state.

We must now determine the solution φ_1 of equation (2.5) that corresponds to the source Y_1. Substituting the latter from equation (2.3.1) into equation (2.2.4) and successively integrating over dx_0', $d\mathbf{r}'$ and dk^0, it is found that

$$\varphi_1(x) = \frac{e}{2\pi^2}\int \exp(i\mathbf{k}.\mathbf{r})\frac{\mathbf{k}.\boldsymbol{\sigma}}{\mathbf{k}^2}\begin{pmatrix}1\\0\end{pmatrix}d\mathbf{k} \tag{2.3.2}$$

where $\mathbf{k}^2 = (k^1)^2 + (k^2)^2 + (k^3)^2$ and $d\mathbf{k} = dk^1 dk^2 dk^3$.

Substituting the Pauli matrices (2.4b) into equation (2.3.2), the solution φ_1 can be expressed as follows

$$\varphi_1 = \frac{e}{2\pi^2}\left\{\begin{pmatrix}0\\1\end{pmatrix}\int \frac{k^1 + ik^2}{\mathbf{k}^2}\exp(i\mathbf{k}.\mathbf{r})\,d\mathbf{k} + \begin{pmatrix}1\\0\end{pmatrix}\int \frac{k^3}{\mathbf{k}^2}\exp(i\mathbf{k}.\mathbf{r})\,d\mathbf{k}\right\} \tag{2.3.3}$$

Integrating by parts, it is readily verified that

$$\int \frac{x_j}{r^3}\exp(-i\mathbf{k}.\mathbf{r})\,d\mathbf{r} = (-ik^j)\frac{4\pi}{\mathbf{k}^2}$$

It then follows from the Fourier transform of this equation that

$$\frac{x_j}{r^3} = \left(\frac{-i}{2\pi^2}\right)\int \frac{k^j}{\mathbf{k}^2}\exp(i\mathbf{k}.\mathbf{r})\,d\mathbf{k} \tag{2.3.4}$$

Substituting equation (2.3.4) into equation (2.3.3), we obtain the following

$$\varphi_1 = -i\frac{e}{r^3}\begin{pmatrix}x_3\\x_1 + ix_2\end{pmatrix} \tag{2.3.5a}$$

Using the same procedure with the second spinor source field Y_2 [equation (2.3.1)], rather that Y_1, the corresponding spinor solution φ_2 of equation (2.5) is found to have the following form:

$$\varphi_2 = \frac{ie}{r^3}\begin{pmatrix}x_1 - ix_2\\-x_3\end{pmatrix} \tag{2.3.5b}$$

3. *Derivation of Coulomb's Law*

The spinor variables in equations (2.3.5a) and (2.3.5b) are the electric field intensity solutions for a static point charge, according to the spinor formulation (2.5) of electromagnetism. These are to be compared with the solutions

$$E_j = ex_j/r^3, \qquad H_j = 0 \qquad (j = 1, 2, 3)$$

of the conventional formalism.

To deduce the mutual forces between point charges, we must now consider the spinor conservation equations (2.1.1). We have seen that one of these can be expressed in the form shown in equation (2.1.4)—in which the *scalar* on the right-hand side plays the role of the time-component of the four-Lorentz force density. Three other conservation equations that depend on the space-components of the Lorentz force density are

$$\partial^k \mathscr{S}(j,k) = \mathscr{F}(j) \qquad (k,j = 1,2,3) \tag{3.1}$$

where

$$\begin{aligned} \mathscr{S}(1,k) &= \left(-\frac{1}{16\pi}\right)(\varphi_1{}^\dagger \sigma_k \varphi_2 + \varphi_2{}^\dagger \sigma_k \varphi_1) \\ \mathscr{S}(2,k) &= \left(\frac{i}{16\pi}\right)(\varphi_2{}^\dagger \sigma_k \varphi_1 - \varphi_1{}^\dagger \sigma_k \varphi_2) \\ \mathscr{S}(3,k) &= \left(-\frac{1}{16\pi}\right)(\varphi_2{}^\dagger \sigma_k \varphi_2 - \varphi_1{}^\dagger \sigma_k \varphi_1) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \mathscr{F}(1) &= \left(-\frac{1}{16\pi}\right)[(\varphi_2{}^\dagger Y_1 + \varphi_1{}^\dagger Y_2) + \text{h.c.}] \\ \mathscr{F}(2) &= \left(\frac{i}{16\pi}\right)[(\varphi_2{}^\dagger Y_1 - \varphi_1{}^\dagger Y_2) + \text{h.c.}] \\ \mathscr{F}(3) &= \left(-\frac{1}{16\pi}\right)[(\varphi_2{}^\dagger Y_2 - \varphi_1{}^\dagger Y_1) + \text{h.c.}] \end{aligned} \tag{3.3}$$

(h.c. ≡ hermitian conjugate).

Let us consider the first of these force terms, $\mathscr{F}(1)$, for the problem at hand. Here,

$$\mathscr{F}(1) = \left(-\frac{1}{16\pi}\right)(\varphi_2^{(m)\dagger} Y_1^{(n)} + \varphi_1^{(m)\dagger} Y_2^{(n)}) + \text{h.c.} \tag{3.4}$$

where (m) refers to the point charge that exerts the electric field $\varphi\alpha^{(m)}$ and (n) refers to the body that is acted upon. Taking the latter to be a point charge of quantity q, located at $\mathbf{r} = \mathbf{a}$, the source field solutions take the form

$$Y_1^{(n)} = -4\pi iq\delta(\mathbf{r}-\mathbf{a})\begin{pmatrix}1\\0\end{pmatrix}, \qquad Y_2^{(n)} = -4\pi iq\delta(\mathbf{r}-\mathbf{a})\begin{pmatrix}0\\1\end{pmatrix} \tag{3.5}$$

The field intensity $\varphi_\alpha^{(m)}$ for a point charge e at the origin is given in equations (2.3.5a) and (2.3.5b). Thus, inserting equations (2.3.5a), (2.3.5b) and (3.5) into equation (3.4), the force density $\mathscr{F}(1)$ takes the form

$$\begin{aligned} \mathscr{F}(1) &= \frac{qe}{2r^3}\left\{(x_1 + ix_2 \quad -x_3)\begin{pmatrix}1\\0\end{pmatrix} + (x_3 \quad x_1 - ix_2)\begin{pmatrix}0\\1\end{pmatrix}\right\}\delta(\mathbf{r}-\mathbf{a}) \\ &= qe\frac{x_1}{r^3}\delta(\mathbf{r}-\mathbf{a}) \end{aligned} \tag{3.6}$$

It then follows that the integrated force that a charge e, at the origin, exerts on a charge q, at $\mathbf{a}$, is

$$F(1) = \int \mathscr{F}(1)\, d\mathbf{r} = qe \int \frac{x_1}{r^3}\delta(\mathbf{r}-\mathbf{a})\, d\mathbf{r} = qea_1/a^3 \tag{3.7}$$

In a similar fashion, it is found that the force densities $\mathscr{F}(2)$ and $\mathscr{F}(3)$ in equation (3.3) give the integrated force terms

$$F(2) = qe(a_2/a^3), \qquad F(3) = qe(a_3/a^3) \tag{3.8}$$

Thus we see that the three *scalars*, $F(j)$, which are derived from the conservation laws (2.1.1) of the factorized spinor equations (2.5) of electromagnetic theory, correspond exactly with the three vector components F_j that represent the Coulomb force that is exerted by a charge e at the origin on another charge q which is separated from it by a distance $a = (a_1{}^2 + a_2{}^2 + a_3{}^2)^{1/2}$.

Of course, the relative orientation of the line of centers between these charged bodies is relative to an arbitrary choice of the orientation of the coordinate frame. With any choice of orientation, one must measure a maximum of three numbers to completely specify the resultant force. The three numbers are identified in the conventional description with the three components of a vector—because, *experimentally*, the sum of their squares is equal to the square of the resultant force (which one may determine with one measurement by specifying a coordinate frame with one of its axes along the line of centers of the interacting charges). Nevertheless, these three numbers, that one identifies with the measurements, is associated here with three scalars. It just turns out in this case, with the approximations that have been made, that each of these sets of three numbers are in one-to-one correspondence. Still, the mathematical formalisms that led to identical predictions *in this application*, are quite different.

The first difference that should be noted is the way in which the 'test charge' appears in the two formulations. In the conventional description, the test charge is introduced by *multiplying* the scalar charge density $q\delta(\mathbf{r} - \mathbf{a})$ by the force field $\mathbf{E}$. In the spinor formulation, the test charge is introduced not as a scalar, but rather as a spinor variable Υ_α. Indeed, it is only when the spinor field Υ_α is combined with the spinor field solution φ_α in the form of an hermitian product, $\varphi_\alpha{}^\dagger \Upsilon_\beta$, that the mutual force can be defined. It is the fact that in the vector formulation one can express the actual force as the product of a field intensity $\mathbf{E}$ and a scalar charge q that leads to the illusory conclusion that one observes the field $\mathbf{E}$, itself. For example, one claims to measure the voltage of a battery where he actually measures the energy difference across the poles of the battery (electron-volts) and then *deduces* the voltage by dividing through by the unit of charge. This is a trivial matter in this case, since the unit of charge is simply a number. However, as we have seen in the above example, the extrapolation from the measured numbers to the vector form of the electric field intensity is not unique.

If one can deduce from the measured numbers two different expressions for the electric field intensity—the vector or the spinor forms—further criteria must then be used to select one of them in the general expression of the theory. The approach that has been taken is that such a criterion is the one of maximum generality—for here one makes the maximum number of predictions of the theory. The spinor formalism for electromagnetism uniquely fulfills this criterion. In the applications considered in this paper the predictions of the two formalisms are identical. But in other applications, referred to earlier (Sachs & Schwebel, 1961, 1963; Sachs, 1968a, b), the spinor formulation predicts effects that are not in the domain of the vector representation of electromagnetism.

4. *Derivation of the Measured Velocity of a Point Charge*

According to the conservation equation (2.1.4),

$$\mathscr{F}(0) = \left(-\frac{1}{16\pi}\right)[(\varphi_1{}^\dagger Y_1 + \varphi_2{}^\dagger Y_2 + \text{h.c.}] \tag{4.1}$$

is the density of power that is dissipated during the flow of electric current. It is the term that corresponds to $\mathbf{E}.\mathbf{j}$ in Poynting's equation.

If $E(\mathbf{r},t)$ is the deduced magnitude of the electric field intensity which is associated with the measured external voltage that causes the current flow, then the magnitude of the resulting current density is the following

$$j = \mathscr{F}(0)/E(\mathbf{r},t) \tag{4.2}$$

Describing this current density in terms of the flow of q units of electric charge per square centimeter per second, the expression from this theory for the measured velocity to be associated with the current density j is the following

$$v = (1/q)\int j\,d\mathbf{r} = \int [\mathscr{F}(0)/qE]\,d\mathbf{r} \tag{4.3}$$

It is important to note that the charge q never appears by itself in any predicted value for a measured property of the system. It rather appears as a factor in a term which is a measure of the coupling of the external electric field and the charged matter that is acted upon.

The expression (4.3) is the required prediction for the velocity of a charged particle, according to the spinor formalism. To obtain an explicit result, one must solve the spinor field equations (2.5) for any special sort of charged system that may be under study. One can then evaluate $\mathscr{F}(0)$ and integrate (4.3) to obtain the final result.

If $\mathscr{F}(0)$ and E should both be time-independent, then one more integration of equation (4.3) gives the following prediction for the measured displacement of the charged matter

$$x - x_0 = \left(\int \mathscr{F}(0)/qE(r)\,d\mathbf{r}\right)(t - t_0) \tag{4.4}$$

where (x_0, t_0) are the position and time measure when the external field is first applied.

4.1. *Application to a Constant External Electric Field*

To check this result with a simple example, suppose that the external field is the constant E, oriented in the x_3-direction, and that the four-current density can be approximated by point charges that move at a constant speed v_0 (along the x_3-direction). In this case, according to equation (2.3),

$$\varphi_1 = iE\begin{pmatrix}1\\0\end{pmatrix}, \qquad \varphi_2 = -iE\begin{pmatrix}0\\1\end{pmatrix}$$

$$Y_1 = -4\pi iev_0\,\delta(\mathbf{r})\begin{pmatrix}1\\0\end{pmatrix}, \qquad Y_2 = 4\pi iev_0\,\delta(\mathbf{r})\begin{pmatrix}0\\1\end{pmatrix} \tag{4.1.1}$$

Substituting these variables into equation (4.1),

$$\begin{aligned}\mathscr{F}(0) &= \left(-\frac{1}{16\pi}\right)\left\{(-4\pi)qv_0\,E\delta(\mathbf{r})\left[(1\quad 0)\begin{pmatrix}1\\0\end{pmatrix} + (0\quad 1)\begin{pmatrix}0\\1\end{pmatrix} + \text{h.c.}\right]\right\}\\ &= (qE)\,v_0\,\delta(\mathbf{r})\end{aligned}$$

Thus, equation (4.3), for the speed of the charged particle, becomes

$$v = (qEv_0/qE)\int \delta(\mathbf{r})\,d\mathbf{r} = v_0$$

as it should. According to equation (4.4), the displacement then takes the expected form

$$x - x_0 = v_0(t - t_0)$$

5. *Summary*

The purpose of the theoretical development in this paper has been analogous to a study of the classical limit of wave mechanics. It is based on the requirement that a mathematical limit must exist in any generalized theory which corresponds to the part of the earlier formalism that has been empirically correct in describing the data. Thus wave mechanics must approach the formalism of the Hamilton–Jacobi theory in mechanics, in the appropriate limit of macroscopic physics—in the sense that the predictions of quantum and classical mechanics must be the same in this limit. Similarly, the factorized spinor formulation of electromagnetic theory must make the same predictions as the vector representation of the theory in the appropriate domain where the extra predictions of the spinor theory are ineffective.

Thus, the attempt has been made in this paper to further elucidate the first-rank spinor formulation of electromagnetism by deriving those solutions that relate to physical effects which agree with the predictions of the standard formalism. One reason for doing this is to emphasize the assertion of this theory that one need not deal with vector or tensor variables in order to predict any of the measured electromagnetic effects—nor does one have to introduce electric charge in terms of a scalar; in this formulation it appears rather as a spinor. One of the main reasons for this contention is that the actual predictions of the theory come from the conservation equations rather than the field equations themselves. The latter part of the formalism must provide the solutions which are the necessary input for the conservation equations. This is analogous to the feature of quantum mechanics that the solutions of the Schrödinger equation are not directly observable, but are rather the necessary input for other mathematical forms, which depend on these solutions, and do relate to the actual measurements.

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